Contest # 1 — *Answers & Solutions* — 10/25/05

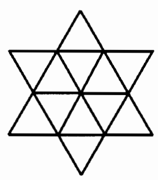
Problem 1-1

$3^2+4^2 \;+\; 5^2+12^2 = 5^2+13^2$, so $a+b = \boxed{18}$.

Problem 1-2

If Roz plans to scare n people, then Sam plans to scare $3n$ people and Tom plans to scare double that, $6n$. Altogether, $n+3n+6n \le 2005$; so $n \le 200.5$. Since n is an integer, n is at most 200. Since Sam plans to scare $3n$ people, that's at most $\boxed{600}$ people.

Problem 1-3



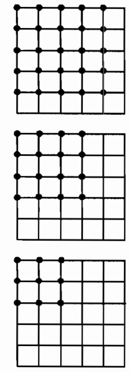
Method I: Draw the 3 diagonals of the hexagon, as shown, to partition the figure into 12 congruent small equilateral triangles. Since the overlap consists of 6 of these triangles, with a total area of 60, each small equilateral triangle has an area of 10. Since each original (large) equilateral triangle consists of 9 small ones, the area of one large equilateral triangle is $\boxed{90}$.

Method II: Join the common center to any two consecutive vertices of the hexagon. The equilateral triangle inside the hexagon is congruent to each equilateral triangle outside the hexagon. Since six of these "inside" triangles make up the hexagon, a large equilateral triangle consists of these six triangles plus three additional triangles. Therefore, the area of a large equilateral triangle is $60 + (1/2)(60) = 90$.

Problem 1-4

If the integer n is greater than $9^2 = 81$ but less than $11^2 = 121$, then $\sqrt{n}$ differs from 10 by less than 1. The set $\{82, 83, \ldots, 119, 120\}$ contains $\boxed{39}$ different integers.

Problem 1-5



Each 1×1 square is determined by its upper left vertex, for which there are 5×5 choices. Each 2×2 square is determined by its upper left vertex, for which there are 4×4 choices. Each 3×3 square is determined by its upper left vertex, for which there are 3×3 choices. In general, each $d\times d$ square is determined by its upper left vertex, for which there are $(6-d)(6-d) = (6-d)^2$ choices. Thus, the number of 1×1 squares is 5^2, the number of 2×2 squares is $4^2, \ldots$, and the number of 5×5 squares is 1^2. The total of the number of squares of all sizes is $5^2+4^2+3^2+2^2+1^2 = \boxed{55}$.

Problem 1-6

Method I: The six possible sums are $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, and $c+d$. Since $a < b < c < d$, the smallest sums are $a+b = 1$ and $a+c = 2$. From these, $c = b+1$. Of the other four sums, the largest are $b+d$ and $c+d$. Of the remaining sums, $b+c$ and $a+d$, one equals 3 and the other equals 4. Since $c = b+1$, if $b+c = 2b+1 = 3$, then $b = 1$. Then, since $a+b = 1$, $a = 0$; and since $a+d = 4$, $d = 4$. If, instead, $b+c = 2b+1 = 4$, then $b = 3/2$. Then, since $a+b = 1$, $a = -1/2$; and since $a+d = 3$, $d = 7/2$. Finally, the two possible values of d are $\boxed{7/2,\ 4}$.

Method II: The six possible sums are $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, and $c+d$. Since $a < b < c < d$, the smallest two sums are $a+b = 1$ and $a+c = 2$. From these, $b = 1-a$ and $c = 2-a$. Of the other four sums, the largest are $b+d$ and $c+d$. Of the remaining sums, $b+c$ and $a+d$, one equals 3, the other equals 4. If $b+c = 3$, then $(1-a)+(2-a) = 3$. Solving, $a = 0$. Since $a+d = 4$, $d = 4$. If $b+c = 4$ and $a+d = 3$, then $(1-a)+(2-a) = 4$, so $a = -1/2$ and $d = 7/2$.

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